NASA CONTRACTOR REPORT

NASA CR-1

NASA CR-192

DERIVATION OF MAPPING FUNCTIONS FOR STAR-SHAPED REGIONS

by Kwan Rim and Roger O. Stafford

Prepared under Grant No. NsG-576 *by*
UNIVERSITY OF IOWA
Iowa City, Iowa
for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1965

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For sale by the Office of Technical Services, Department of Commerce,
Washington, D.C. 20230 -- Price $2.00

ABSTRACT

DERIVATION OF MAPPING FUNCTIONS

FOR STAR-SHAPED REGIONS

This report presents a simple method of deriving approximate mapping functions in the form of low order polynomials, which conformally map an annular region onto one whose inner and outer boundaries are star shaped and circular. The derivation is based on the Schwarz-Christoffel transformation.

Illustrations are carried out in detail for a number of typical star-shaped regions. Some of the final results are presented by graphs, from which one may readily derive a particular polynomial for the approximate mapping of a certain class of star-shaped regions.

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DERIVATION OF MAPPING FUNCTIONS FOR STAR-SHAPED REGIONS

By Kwan Rim and Roger O. Stafford

Department of Mechanics & Hydraulics
The University of Iowa
Iowa City, Iowa

SUMMARY

The purpose of this report is to present a simple method of deriving the approximate mapping functions in the form of low order polynomials, which transact the conformal mapping of annular regions onto those whose inner and outer boundaries are star-shaped and circular. Since the entire derivation is based on the well-known Schwarz-Christoffel transformation, its application to a practical problem is simple and straight forward. Compared to other approximate methods of conformal mapping, the present method does provide simpler mapping functions; namely, polynomials with a smaller number of terms.

Illustrations are carried out in detail for a number of typical star-shaped regions. Some of the final results are presented by graphs, from which one may readily derive a particular polynomial for the approximate mapping of a certain class of star-shaped regions.

INTRODUCTION

The main objective of this research project is to perform the elastic and viscoelastic analysis of two-dimensional problems with star-shaped boundaries by the method of complex variables. A successful application of this method requires the derivation of a mapping function which maps a given region conformally onto a unit circle. Hence, the first phase of the research, covered by this First Interim Report, is devoted to the development of a simple method of deriving a satisfactory mapping function.

Since the exact mapping functions are not derivable for many physical problems, a practical alternative is to make use of an approximate mapping function which accomplishes the conformal mapping of a unit circle onto a region reasonably congruent to a given star-shaped domain. The desirable characteristics of such an approximate mapping function are: the reasonable congruency of the mapped to the specified region and the simplicity of the mapping function. Since these two characteristics are usually incompatible with each other, one has to seek a proper compromise for a given problem.

The significance of the congruency is self-evident and does not require any further explanation. The simplicity is also very important because a complicated mapping function tends to diminish the merits of analytic solutions. As the complexity of an analytic solution increases, so does the difficulty in comprehending its physical implications. It is also to be noted that the numerical answers provided by a complicated analytical solution may be obtained by some reliable numerical methods with less computational effort.

A comprehensive treatment of approximate conformal mapping may be found in the book by Kantorovich and Krylov (ref. 1). Numerous investigators (ref. 2 and 3), who have utilized the various methods of approximate mapping, have reported that a polynomial with a very large number of terms is required to accomplish a satisfactory mapping of a star-shaped region. The number of terms runs from several dozen to a few hundred. For example, Parr (ref. 3) reported that a polynomial with fifty terms was required to achieve a satisfactory mapping of a fairly simple star.

The authors have found that a proper application of the well-known Schwarz-Christoffel transformation to the star-shaped regions offers several advantages. The first advantage is that this method is simple and may be readily understood by an average analyst. The coefficients of the power series are automatically determined and remain fixed, and they are related to the physical parameters which control the size and shape of a star. The second advantage is that it yields polynomials with a considerably less number of terms for certain kinds of stars. This can be accomplished by examining the congruency and truncating the series according to a desired accuracy. Another point of great significance is that all the polynomials obtained through truncation do satisfy the condition of conformality automatically.

Most of the other methods of approximate conformal mapping do not possess these properties. In many cases, the coefficients of a polynomial mapping function must be re-evaluated for every new approximation and some methods require one to start with a polynomial with an enormously large number of terms in order to ensure that the condition of conformality will be satisfied.

For a clear illustration of the method, a simple star with four points is chosen and the procedure is carried out in detail. Although it is simple, it contains all the essential features. Some of the final results are presented in graphical form, which will allow one to select the proper polynomial for the approximate mapping of a given class of stars. Besides this example, a number of other stars have been investigated, and useful results are briefly presented at the end.

SYMBOLS

i	$\sqrt{-1}$
z, ζ	Complex numbers; i.e., $z = x + iy$
$f(\zeta)$	Mapping function
$\bar{z}, \bar{\zeta}$	Conjugates of complex numbers; i.e., $\bar{z} = x - iy$
m	Number of star points
j, k, l	Indices
K_j	Exterior angle of the j-th vertex divided by π
n	Degree of a star
$a_j, b_j, \ldots$	Images of vertices on the unit circle
γ_j	Spacings of images on the unit circle
F	Polynomials of ζ
S_k	Distance between adjacent vertices of a star

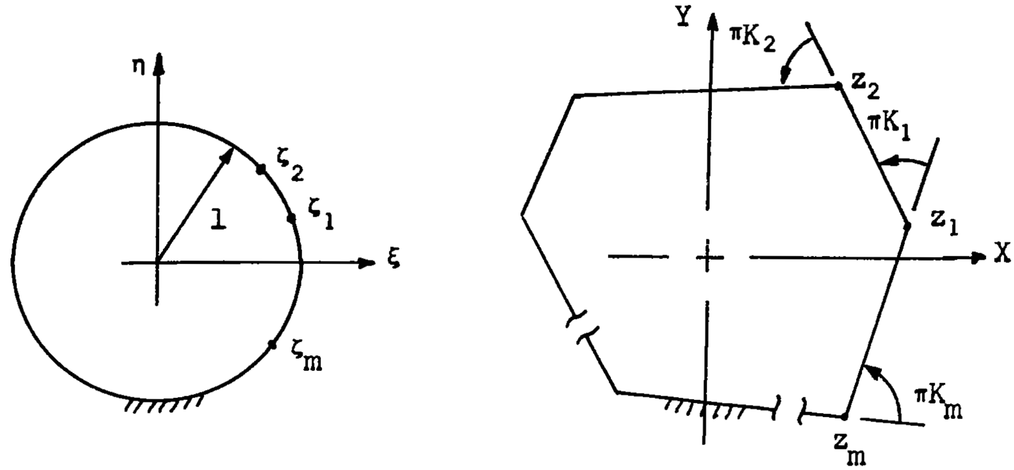
DERIVATION OF MAPPING FUNCTIONS

The conformal mapping of the exterior of a unit circle onto the exterior of a closed polygon is accomplished by an application of the Schwarz-Christoffel transformation and may be transacted by the following general formula (ref. 4):

$$z = f(\zeta) = A \int_0^{1/\zeta} (\zeta_1 - \zeta)^{K_1} (\zeta_2 - \zeta)^{K_2} \ldots (\zeta_m - \zeta)^{K_m} \frac{d\zeta}{\zeta^2} \tag{1}$$

in which A is a complex constant and K_j is related to the exterior angle of the polygon at the vertex z_j by the factor of π. The sum of a polygon's exterior angles being 2π, the summation of K_j is always two. Refer to figure 1 for other details.

The applicability of the Schwarz-Christoffel transformation is supported by the fact that any geometric figure may be closely approximated by a polygon and that the symmetric nature of a star-shaped region greatly simplifies the final form of the mapping function. Hence, the use of this method involves an



Unit circle on ζ-plane

Polygon on z-plane

Figure 1.-Mapping of the exterior of a unit circle onto the exterior of a polygon.

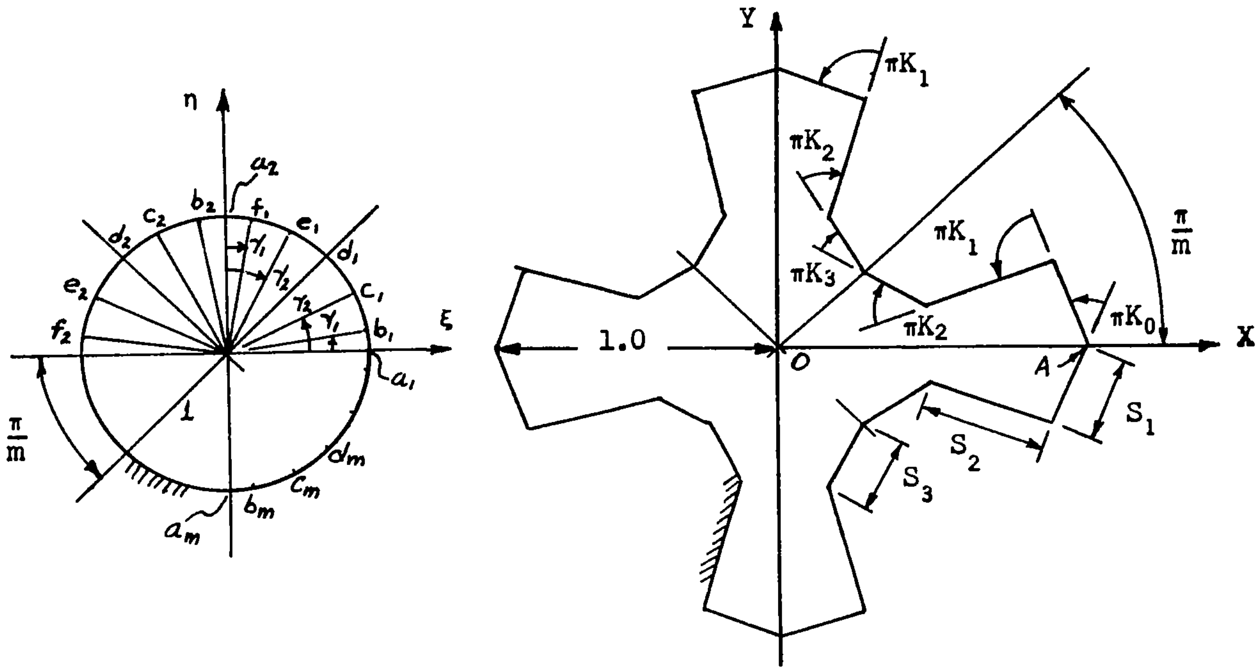
approximation of an actually specified region by a polygon. The approximating polygon is then mapped onto a unit circle by the well-established method of the Schwarz-Christoffel transformation. The details of this procedure and the reduction of the general mapping formula will be shown in the derivation of the mapping function for a specific example.

Consider a group of stars which may be well approximated by the type of polygon shown in figure 2. The mapping function for this kind of polygon is sufficiently simple so as to facilitate a lucid illustration of the method, yet it retains the basic characteristics of a general polygon. The illustration will be carried out in such a way that one will be able to derive an approximate mapping function for an arbitrary star. For brevity, the word "star" will be used in place of "star-shaped polygon."

Proper substitution and regrouping of terms in equation (1) reduces the mapping function to

$$z = A \int_0^{1/\zeta} \frac{[(a_1 - \zeta)\ldots(a_m - \zeta)]^{K_0} \cdot [(b_1 - \zeta)\ldots(b_m - \zeta)]^{K_1}}{[(c_1 - \zeta)\ldots(c_m - \zeta)]^{K_2} \cdot [(d_1 - \zeta)\ldots(d_m - \zeta)]^{K_3}}$$

$$\times \frac{[(f_1 - \zeta)\ldots(f_m - \zeta)]^{K_1}}{[(e_1 - \zeta)\ldots(e_m - \zeta)]^{K_2}} \frac{d\zeta}{\zeta^2} \qquad (2)$$



Unit circle on ζ-plane

Star-shaped polygon on z-plane

Figure 2.- Mapping of a polygonal star; m = number of star points, positive K_j is counter clockwise.

in which a_j, b_j, ... f_j are defined by

$$a_1 = e^{i \cdot 0}, \quad b_1 = e^{i\gamma_1}, \quad c_1 = e^{i\gamma_2}, \quad d_1 = e^{i\frac{\pi}{m}}, \quad e_1 = e^{i(\frac{2\pi}{m} - \gamma_2)},$$

$$f_1 = e^{i(\frac{2\pi}{m} - \gamma_1)};$$

and by the general relationship

$$\beta_j = \beta_1 \cdot e^{i\frac{2\pi}{m}(j-1)}; \qquad \beta_j = a_j, b_j, c_j, d_j, e_j, f_j;$$
$$(j = 1, 2, \dots m).$$

Intermediate steps of derivation are relatively simple; therefore, they are omitted for the sake of concise presentation.

It can be shown that

$$(\beta_1 - \zeta)(\beta_2 - \zeta) \dots (\beta_m - \zeta) = \prod_{j=1}^{m} [\beta_1 e^{i \frac{2\pi}{m} (j-1)}] = \beta_1^m - \zeta^m.$$

Then by observing that a_m and d_m correspond to the m-th roots of +1 and -1 respectively, equation (2) reduces to

$$z = A \cdot \int_0^{1/\zeta} \frac{(1 - \zeta^m)^{K_0} \vdots [(b_1^m - \zeta^m) \cdot (f_1^m - \zeta^m)]^{K_1}}{(1 - \zeta^m)^{K_3} \vdots [(c_1^m - \zeta^m) \cdot (e_1^m - \zeta^m)]^{K_2}} \cdot \frac{d\zeta}{\zeta^2}$$

$$= \alpha' \int_0^{1/\zeta} \frac{(1 - \zeta^m)^{K_0} \vdots [(1 - b_1^{-m}\zeta^m)(1 - f_1^{-m}\zeta^m)]^{K_1}}{(1 - \zeta^m)^{K_3} \vdots [(1 - c_1^{-m}\zeta^m)(1 - e_1^{-m}\zeta^m)]^{K_2}} \cdot \frac{d\zeta}{\zeta^2},$$

and finally to

$$z = \alpha'' \int_0^{1/\zeta} \frac{(1 - \zeta^m)^{K_0} \vdots [(1 - e^{-im\gamma_1}\zeta^m)(1 - e^{im\gamma_1}\zeta^m)]^{K_1}}{(1 - \zeta^m)^{K_3} \vdots [(1 - e^{-im\gamma_2}\zeta^m)(1 - e^{im\gamma_2}\zeta^m)]^{K_2}} \frac{d\zeta}{\zeta^2}. \qquad (3)$$

This integral, equation (3), cannot be evaluated in terms of elementary functions. But a solution in the form of a power series can be obtained by expanding the integrand into a power series and integrating it term by term. Expansion of the integrand is carried out by performing the binomial expansion of each parenthesized quantity and then forming the product of all the series. A general series associated with the j-th vertex can be developed by expanding the quantity

$$[(1 - e^{-im\gamma_j}\zeta^m)(1 - e^{im\gamma_j}\zeta^m)]^{K_j}$$

in the following manner:

$$(1 - e^{-im\gamma_j}\zeta^m)^{K_j} = 1 + \sum_{k=1}^{\infty} \left[\prod_{n=0}^{k-1} (K_j - n)\right] \frac{e^{-im\gamma_j k} (-\zeta^m)^k}{k!} = \sum_{k=0}^{\infty} A_k \zeta^{km}.$$

$$(1 - e^{im\gamma_j}\zeta^m)^{K_j} = 1 + \sum_{k=1}^{\infty} \left[\prod_{n=0}^{k-1} (K_j - n)\right] \frac{e^{im\gamma_j k} (-\zeta^m)^k}{k!} = \sum_{k=0}^{\infty} B_k \zeta^{km},$$

Hence

$$[(1 - e^{-im\gamma_j} \zeta^m)(1 - e^{im\gamma_j} \zeta^m)]^{K_j} = \sum_{k=0}^{\infty} C_k (\gamma_j, K_j) \cdot \zeta^m$$

$$= F(\zeta, \gamma_j, K_j) \qquad (4)$$

where the coefficients C_k are defined by

$$C_k(\gamma_j, K_j) = \sum_{\ell=0}^{k} A_\ell \cdot B_{k-\ell} \; ;$$

$$C_0(\gamma_j, K_j) = 1 \; ,$$

$$C_1(\gamma_j, K_j) = -k_j \cdot 2 \cos (m\gamma_j) \; ,$$

$$C_2(\gamma_j, K_j) = \frac{K_j(K_j - 1)}{2!} 2 \cos (2m\gamma_j) + K_j \cdot K_j \; ,$$

$$C_3(\gamma_j, K_j) = \frac{-K_j(K_j - 1)(K_j - 2)}{3!} 2 \cos (3m\gamma_j)$$

$$- \frac{K_j}{1!} \frac{K_j(K_j - 1)}{2!} 2 \cos (m\gamma_j) \; ,$$

$$C_4(\gamma_j, K_j) = \frac{K_j(K_j - 1)(K_j - 2)(K_j - 3)}{4!} 2 \cos (4m\gamma_j)$$

$$+ \frac{K_j}{1!} \frac{K_j(K_j - 1)(K_j - 2)}{3!} 2 \cos (2m\gamma_j) + \left[\frac{K_j(K_j - 1)}{2!}\right]^2 .$$

Since the following binomial expansions

$$(1 - \zeta^m)^K = 1 - K\zeta^m + \frac{K(K - 1)}{2!} \zeta^{2m} - \frac{K(K - 1)(K - 2)}{3!} \zeta^{3m} + \dots ,$$

$$(1 + \zeta^m)^{-K} = 1 - K\zeta^m + \frac{K(K + 1)}{2!} \zeta^{2m} - \frac{K(K + 1)(K + 2)}{3!} \zeta^{3m} + \dots ,$$

may be regarded as special limiting cases of equation (4), they may be written as

$$(1 - \zeta^m)^K = \sum_{k=0}^{\infty} C_k(0, \frac{K}{2}) \cdot \zeta^{km} = F(\zeta, 0, \frac{K}{2}) ,$$

$$(1 + \zeta^m)^{-K} = \sum_{k=0}^{\infty} C_k(\frac{\pi}{m}, - \frac{K}{2}) \zeta^{km} = F(\zeta, \frac{\pi}{m}, - \frac{K}{2}).$$

Thus equation (3) may now be expressed in terms of the products of the series defined by equation (4)

$$z = \alpha'' \int_0^{1/\zeta} F(\zeta, 0, \frac{K_0}{2}) \cdot \prod_{j=1}^{n} [F(\zeta, \gamma_j, K_j)] \cdot F(\zeta, \frac{\pi}{m}, \frac{K_{n+1}}{2}) \frac{d\zeta}{\zeta^2} , \tag{5}$$

which becomes

$$z = \alpha'' \int_0^{1/\zeta} [\sum_{k=0}^{\infty} D_k \zeta^{km-2}] \, d\zeta . \tag{6}$$

In equation (6), n is the number of vertices between two adjacent lines of symmetry, and the series $D_k \zeta^{km}$ is the final product of the series in the integrand of equation (5). By integrating the series term by term, one obtains the mapping function for the type of star shown in figure 2 in the following form:

$$z = f(\zeta) = \alpha \sum_{k=0}^{\infty} \frac{D_k \zeta^{1-km}}{1 - km} . \tag{7}$$

Since the use of such a mapping function expressed in an infinite series is prohibitive, a polynomial mapping function which is derived by truncating the infinite series is used in a practical application.

The behavior of the mapping function defined by equation (7) is controlled by three types of parameters. They are the normalizing coefficient α, the vertex angles (K_1 and K_2), and the spacing of the images of vertices on the unit circle (γ_1 and γ_2). In order to describe the behavior of the mapping function, the role of each type of parameter will be discussed in the following pages.

The mapping functions for other types of stars, such as shown in figure 3, can be derived in a similar manner as equation (7) was obtained. Therefore, no further elaboration will be made on other types of stars, except for brief graphical presentations of interesting mappings obtained for such stars. See the figures presented at the end of the report.

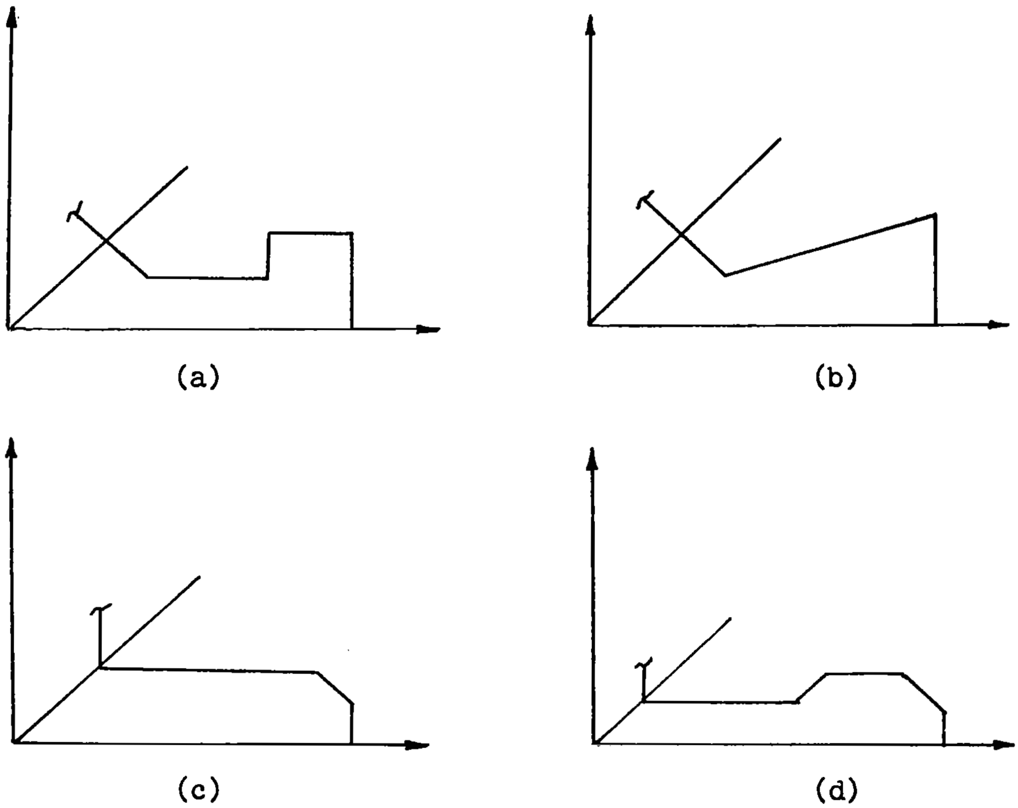


Figure 3.- Segments of various four-pointed stars.

PARAMETERS CONTROLLING THE SIZE AND SHAPE

The normalizing coefficient α is determined in such a way that the distance from the center to the outer tip of a star is a unity and that one of the star's major axes of symmetry is oriented along the x-axis. It is readily evaluated by letting $z = 1$ be the image of $\zeta = 1$; i.e.

$$1 = \alpha \sum_{k=0}^{\infty} \frac{D_k}{1 - km} .$$

As a convenient means of classifying various stars, the term "degree" will be used hereafter. The degree of a star corresponds to the number of vertices counted between any two adjacent axes of symmetry, excluding the ones located on the axes. An m-pointed star possesses $2m$ axes of symmetry and an n-th degree star has $(n + 1)$ straight sides between adjacent axes of symmetry. For instance, the four-pointed star shown in figure 2 is a second degree star and has three straight line segments between adjacent axes of symmetry; namely, S_1, S_2 and S_3. It has of course eight axes of symmetry.

The shape of a star is determined by its exterior vertex angles and the relative dimensions of its sides. Except for the case of a zero degree star, the angles alone are not sufficient to describe the configuration of a star. In general the angles are specified in terms of K_j and the relative dimensions of the sides are controlled by γ_j —the position of the vertex images on the unit circle.

Due to its symmetric property, a star can be uniquely defined by specifying its configuration only in a sector bounded by two adjacent axes of symmetry. The angle between adjacent axes of symmetry is π/m radians and it is preserved in the course of the conformal transformation. Therefore the mapping function for an n-th degree star is a function of only K_j and γ_j $(j = 1, 2, \ldots, n)$ determined from the small sector of the star. It is due to this fact that the form of the general mapping formula is greatly simplified in the case of a star, as it has been illustrated in the preceding derivation.

The number of independent side dimensions S_k equals the degree of a star; that is, for an n-th degree star only n of S_k are independent, even though there are $(n + 1)$ of S_k between adjacent axes of symmetry. It has been observed that S_k may be approximately expressed in terms of γ_j by the following relationship:

$$S_k \approx \frac{\gamma_k}{\left(\frac{\pi}{m} - \sum_{j=1}^{k-1} \gamma_j\right)}, \quad k = 1, 2, \ldots n. \tag{9}$$

It is very little affected by γ_j if $j > k$. For any particular type of a star, a more exact relationship between S_k and γ_j $(j = 1, 2, \ldots, n)$ can be made available in the form of graphs. These graphs will allow one to determine γ_k in an expedient manner for a high degree of accuracy. As an example, these graphical relations for a second-degree star are presented in Appendix A, together with an explanation of the usage.

EFFECT OF TRUNCATION

The general mapping formula for a polygon with k vertices is expressible in the following form:

$$z = f(\zeta) = A \int (\zeta - \zeta_1)^{K_1} (\zeta - \zeta_2)^{K_2} \ldots (\zeta - \zeta_k)^{K_k} \frac{d\zeta}{\zeta^2} + B \tag{10}$$

in which the j-th binomial is associated with the j-th vertex and in fact it plays the major role of controlling the mapping in the vicinity of that vertex. This may be readily seen from an inspection of the derivative of the mapping function (10),

$$f'(\zeta) = A\zeta^{-2} (\zeta - \zeta_1)^{K_1} (\zeta - \zeta_2)^{K_2} \ldots (\zeta - \zeta_k)^{K_k} .$$

The argument of the derivative is the sum of the arguments of the individual terms;

$$\text{Arg } f'(\zeta) = \text{Arg } (A\zeta^{-2}) + K_1 \text{ Arg } (\zeta - \zeta_1) + K_2 \text{ Arg } (\zeta - \zeta_2) + \ldots .$$

If we let ζ vary from $\zeta_j - \varepsilon$ to $\zeta_j + \varepsilon$, $(1 < j < k,\ \varepsilon << 1)$; then the magnitude of the change in the argument of the j-th binomial $(\zeta - \zeta_j)^{K_j}$ is $K_j \pi$, while that of any other binomial is less than $K_\ell \pi \varepsilon$, $(\ell \neq j)$.

The polynomial for the j-th binomial $(\zeta - \zeta_j)^{K_j}$, which is obtained by expanding $(\zeta - \zeta_j)^{K_j}$ into a truncated power series, is evidently a very rapidly varying function in the vicinity of the j-th vertex. On the other hand, the polynomials corresponding to the other binomials are all slowly varying functions in the same vicinity. Therefore, the accuracy of mapping in the vicinity of the j-th vertex depends primarily on the number of terms retained in the polynomial for $(\zeta - \zeta_j)^{K_j}$, and very little on the other polynomials. The more terms retained in a particular polynomial, the more accurate the mapping of the corresponding vertex region becomes. Hence, with regard to the effect of truncating the power series for the j-th binomial, the following remark is in order:

<u>Remark I</u>: Mapping of the vicinity of the j-th vertex is primarily controlled by the j-th binomial; hence the accuracy of the mapping function in that vicinity depends on the accuracy (the extent of truncation) of the corresponding polynomial, e.g., equation (4).

The exact number of terms to be retained in a specific polynomial $F(\zeta, \gamma_j, K_j)$ depends upon the accuracy desired at the j-th vertex and the configuration of the star. This is determined in a practical problem through a trial and error method. However, it has been observed that the polynomial for a vertex with a positive exterior angle is more rapidly converging than that for a vertex with a negative exterior angle. For the same accuracy of mapping, therefore, fewer terms are required in the former polynomial than in the latter. The positive exterior angle is measured in a counter-clockwise direction.

An inspection of the typical series for each case will clearly illustrate this fact. For each case, consider the following typical series:

$$(1 - \zeta)^K = 1 - K\zeta + \frac{K(K-1)}{2!}\zeta^2 - \frac{K(K-1)(K-2)}{3!}\zeta^3 + \ldots \qquad (11)$$

$$(1 + \zeta)^{-K} = 1 - K\zeta + \frac{K(K + 1)}{2!} \zeta^2 - \frac{K(K + 1)(K + 2)}{3!} \zeta^3 + \dots \quad (12)$$

The ratio of the successive terms of each series is given by

$$R_n^+ = \frac{n - K}{n + K} \qquad \text{(for the vertex with a positive exterior angle),}$$

and

$$R_n^- = \frac{n + K}{n + 1} \qquad \text{(for the vertex with a negative exterior angle).}$$

This ratio being a measure of the rate of convergence, we can conclude from

$$R_n^+ < R_n^-$$

that series (11) converges faster than series (12). Hence, the following concluding remark is in order:

Remark II: For the same accuracy of mapping, the polynomial for a vertex with a positive exterior angle requires a lesser number of terms than the polynomial for a vertex with a negative exterior angle.

The geometrical significance of truncation is that of mapping a figure nearly congruent to the prescribed polygon but with rounded corners. In the case of a star, it is observed that the polynomial mapping function maps a unit circle onto a so-called curvilinear polygon.

CONSTRUCTION OF MAPPING FUNCTIONS

A general method of constructing a mapping function for a simple star will be first described. Then it will be shown how this method may be simply extended to the case of a doubly connected star. The mapping of a simple star is to be understood as the conformal mapping of the exterior of the unit circle to the exterior of a star, and that of a doubly connected star as the conformal transformation of an annular region onto a domain whose inner and outer boundaries are star-shaped and circular.

Starting with two sets of known data (K_j and S_j) which are prescribed for any given star, the construction of an approximate mapping function for a simple star is carried out by the following steps:

1. Estimate the image spacing γ_j from the given side dimensions S_j.
2. Generate the polynomials, equation (4), for each vertex; i.e., $F(\zeta; \gamma_j, K_j)$, $j = 0, 1, 2, \dots, n + 1$.
3. Truncate the polynomials according to the accuracy of mapping desired at each vertex.

4. Multiply the polynomials together, integrate the product and then normalize the resulting mapping function.

5. Plot the mapped star and refine the congruency as needed, by modifying the spacing of vertex images and/or varying the number of terms in the polynomials.

Some of the steps (i.e. steps 1, 2, and 4) have been already explained in full detail. The third step must be accomplished by a trial and error method, which is greatly facilitated by the two useful remarks previously stated in connection with the effect of truncation. In fact, for the second degree stars upper and lower bounds on the number of terms have been established. For the maximum error of 5% in the dimensions of S_j, the polynomial for the vertex with a positive exterior angle requires only 4 to 10 terms, and that with a negative angle 5 to 25 terms. That this method does provide a simpler mapping function for a certain class of stars is demonstrated in figure 17; the mapping of a fairly deep star is accomplished by a 12-term polynomial.

The second degree star shown in figure 2 was examined in detail with the aid of a digital computer. Accurate relationships between γ_j and S_j were completely established and are presented by graphs in Appendix A. A working FORTRAN program for the IBM 7040 is now available which automatically generates a mapping function and plots the mapped star. This program automatically determines the coefficients of the polynomials $F(\zeta; \gamma_j, K_j)$ for each vertex from the input data of K_j and γ_j, truncates the polynomials after the prescribed number of terms, produces a mapping function by the step number 4 and plots the mapped star. It takes less than twenty seconds of computing time to construct and plot a mapping function in this manner. Examples of mapped stars, from second to fourth degree, are also presented in Appendix B.

The polynomial mapping function for a simple star may also be used for the mapping of a doubly connected star since the function

$$z = f(\zeta) = \sum_{k=0}^{\infty} E_k \, \zeta^{1-km}$$

reduces to the following equation for $\zeta >> 1$;

$$f(\zeta) \doteq E_0 \cdot \zeta \, .$$

This shows that a circle in ζ-plane whose radius is considerably greater than one is mapped onto a figure in z-plane which is nearly equal to a circle with an approximate radius of $E_0 \mid \zeta \mid$. For a second-degree star with four points, the image of $\mid \zeta \mid = 2$ is found to vary from a circle by not more than 2%. For practical purposes, equation (13) performs a satisfactory mapping of doubly connected stars.

CONCLUDING REMARKS

The success of complex variable methods in elasticity hinges upon the availability and simplicity of a mapping function. Previous investigators (ref. 2 and 3) have derived mapping functions by approximate methods involving some variation of the collocation technique. However, these approaches require very extensive computations and usually yield very lengthy polynomial mapping functions.

Present investigation has attacked the specific problem of mapping a unit circle onto a star-shaped region by the Schwarz-Christoffel transformation. By capitalizing on the high degree of symmetry of star-shaped polygons, a generalized method was developed which yields directly the desired mapping function. Some comparisons have shown that equivalent polynomial mapping functions may be generated with less than one-fourth the number of terms required by the collocation technique.

This new method has two drawbacks, the most important being that for higher degree stars not covered by the figures in Appendix A a high speed digital computer is a prerequisite to the development of satisfactory mapping functions. The second restriction is that the graphical presentation of the relations between γ_k and S_k for third and higher degree stars becomes extremely difficult except for specific cases. However, excellent first approximation to the γ_k can be obtained from the figures presented in Appendix A, and accurate maps of some fifth degree stars have been developed without extra effort.

In general, the Schwarz-Christoffel method developed herein does provide a simpler mapping function for that class of stars which can be approximated by polygons, and it offers some unique advantages. Some of these are: unique and automatic determination of the coefficients of a polynomial from three geometric parameters, automatic satisfaction of the conformality condition, and the fact that the variables have simple geometric interpretations.

Perhaps the chief advantage is that this method will provide the simplest possible polynomial mapping function, as one may easily truncate the polynomial to the lowest number of terms that will accomplish the desired transformation.

APPENDIX A. ANALYSIS OF SECOND-DEGREE STARS

A detailed analysis was performed on a second-degree star with $K_0 = K_3 = 0$, see figure 2. This star has four independent parameters: the number of star points, m; the angle K_1 (note that $K_2 = 1/m - K_1$); and the relative side dimensions S_1 and S_2. The mapping function is given by equation (5);

$$F(\zeta) = \alpha \int_0^{1/\zeta} F(\zeta, \gamma_1, K_1) \cdot F(\zeta, \gamma_2, K_2) \frac{d\zeta}{\zeta^2} .$$

The purpose of this analysis was twofold: (a) To determine the relationships between the relative dimensions S_k and the vertex image spacings γ_k in terms of γ_j, m and K_1; (b) To determine the effects on congruency of truncating the polynomial $F(\zeta, \gamma_j, K_j)$ in order to establish limits of truncation. A program for the IBM 7040 computer was developed to perform these tasks, and the results are shown in the following figures and a table.

The graphs are plots of S_k versus γ_k for different values of m and K_1, and table 1 contains the limits of truncation. The dimensionless parameter S_k is defined as the distance between the (k - 1)-th and k-th vertices divided by the distance $\overline{OA}$ (see figure 2). Only the usage of the graphs will be outlined here; that is, how to determine γ_1 and γ_2 from the given data m, K_1, S_1 and S_2. The procedure may be broken down into the following steps.

(1) From the given values of m and K_1, select the appropriate pair (S_1 and S_2) of graphs. The angles need not agree exactly, but select graphs whose K_1 is nearest the given angle.

Since S_1 is nearly independent of γ_2, the following procedure will quickly yield the correct image spacings, γ_1 and γ_2.

(2) From the graph of S_1 versus $\gamma_1/(\pi/m)$, find the maximum and minimum values of γ_1 for the given S_1 and calculate the average value.

(3) On the graph of S_2 versus $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$, move along the appropriate $\gamma_1/(\pi/m)$ = constant line to the specified S_2 value, and read off the corresponding value of $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$.

(4) On the graph used in the step (2), find a corrected value of γ_1 by moving along the line $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$ = constant (which was found in step 3) to the specified S_1 value, and read off a new value of γ_1.

(5) Repeat steps (2) through (4) until γ_1 and γ_2 converge.

For example, with $m = 4$, $K_1 = .5$, $S_1 = S_2 = .2$, the values found from figures 4 and 5 are $\gamma_1 = 0.395/(\pi/4)$ and $(\gamma_1 - \gamma_2)/(\pi/4 - \gamma_1) = 0.43$.

These charts are very dependent on the number of star points, m, as an increase in m will primarily cause a significant decrease in the average slope of the curves. Also these curves are not independent of K_1, but the variation of K_1 will have a small effect on S_k. It has been observed that for a particular S_1 and S_2, a 24% increase in K_1 will cause an 8% increase in $\gamma_1/(\pi/m)$ and an 8% decrease in $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$.

No. of	$F(\zeta, \gamma_1, K_1)$		$F(\zeta, \gamma_2, K_2)$	
Star Points	S_1	No. of Terms	S_2	No. of Terms
4	$S_1 < .25$	3	$S_2 < .3$	6
	$.25 < S_1 < .5$	4	$.3 < S_2 < .6$	8
6	$S_1 < .1$	3	$S_2 < .3$	6
	$.15 < S_1 < .4$	4	$.3 < S_2 < .7$	9
8	$S_1 < .1$	4	$S_2 < .4$	8
	$.1 < S_1 < .3$	5	$.4 < S_2 < .7$	13

Table 1.-Truncation limits of second degree stars for a maximum error of 5% in S_k.

The following two pages show four examples of the effect of truncation. The first three plots illustrate that a polynomial with a very low number of terms can yield a mapping with congruency. The fourth plot shows that truncation to two terms will cause a smooth rounding of a vertex.

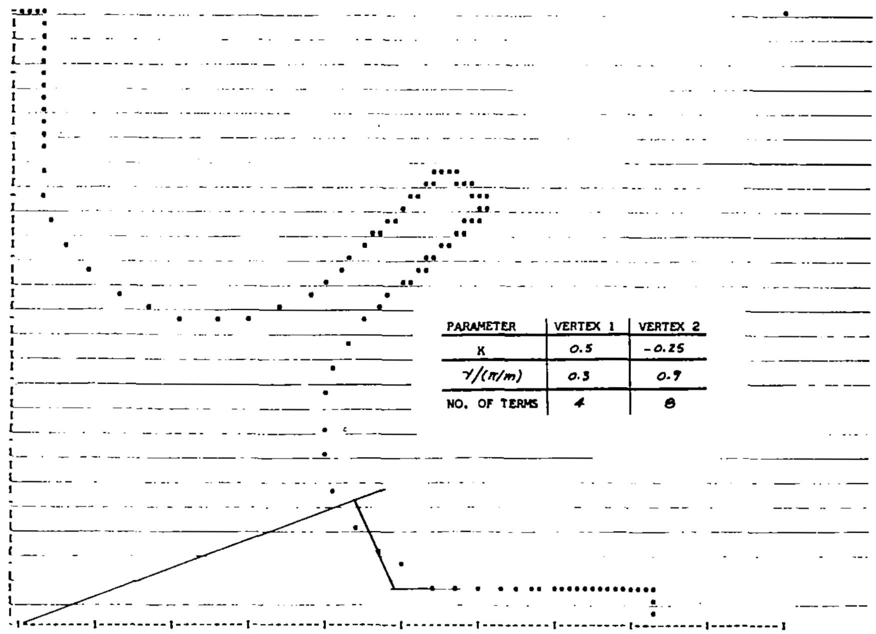


Figure 4.- An 8-point star mapped by a 13-term polynomial.

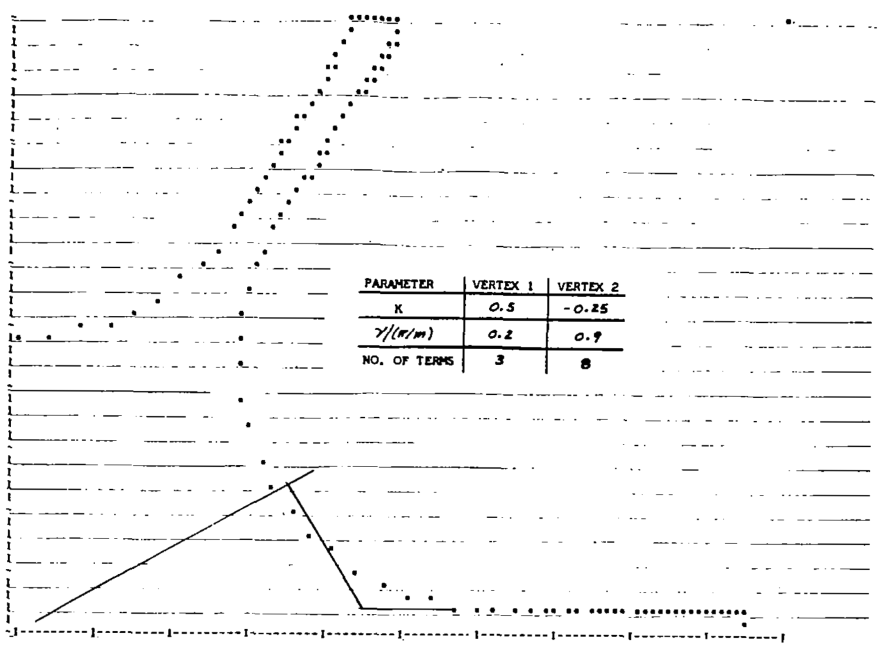
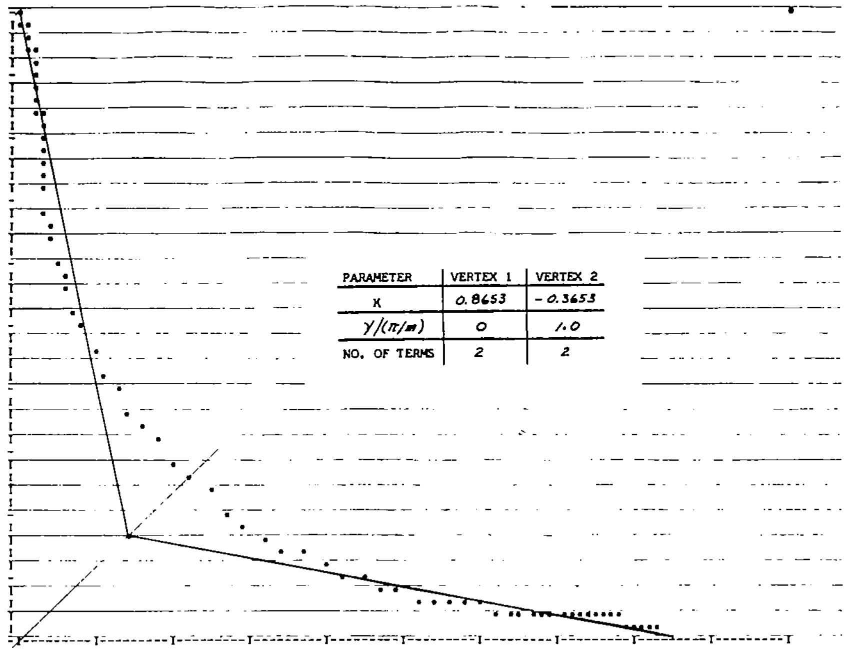
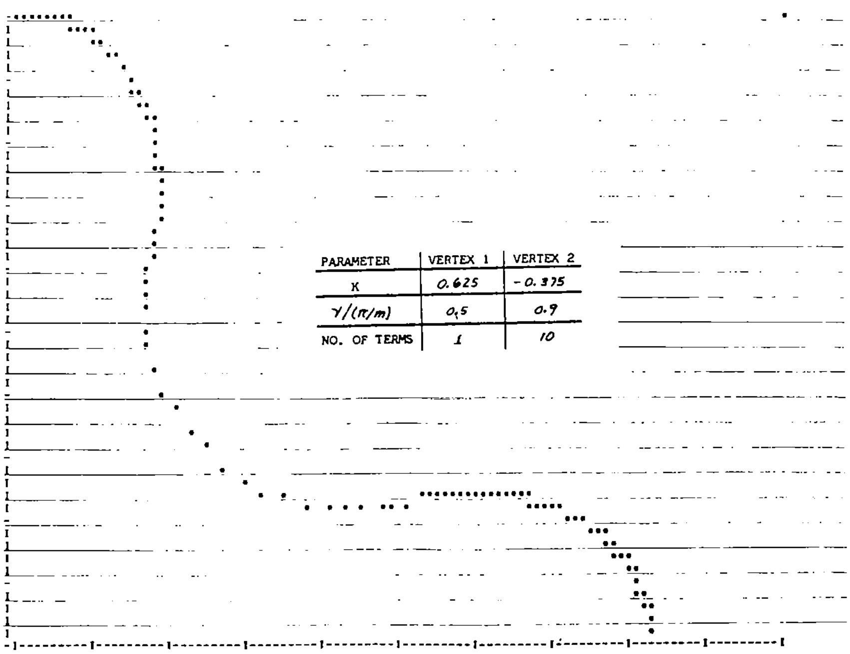


Figure 5.- A 6-point star mapped by a 12-term polynomial.



PARAMETER	VERTEX 1	VERTEX 2
X	0.8653	-0.3653
$\gamma/(\pi/m)$	0	1.0
NO. OF TERMS	2	2

Figure 6.- A 4-point star mapped by a 5-term polynomial.



PARAMETER	VERTEX 1	VERTEX 2
X	0.625	-0.375
$\gamma/(\pi/m)$	0.5	0.9
NO. OF TERMS	1	10

Figure 7.- A 4-point star mapped by a 12-term polynomial.

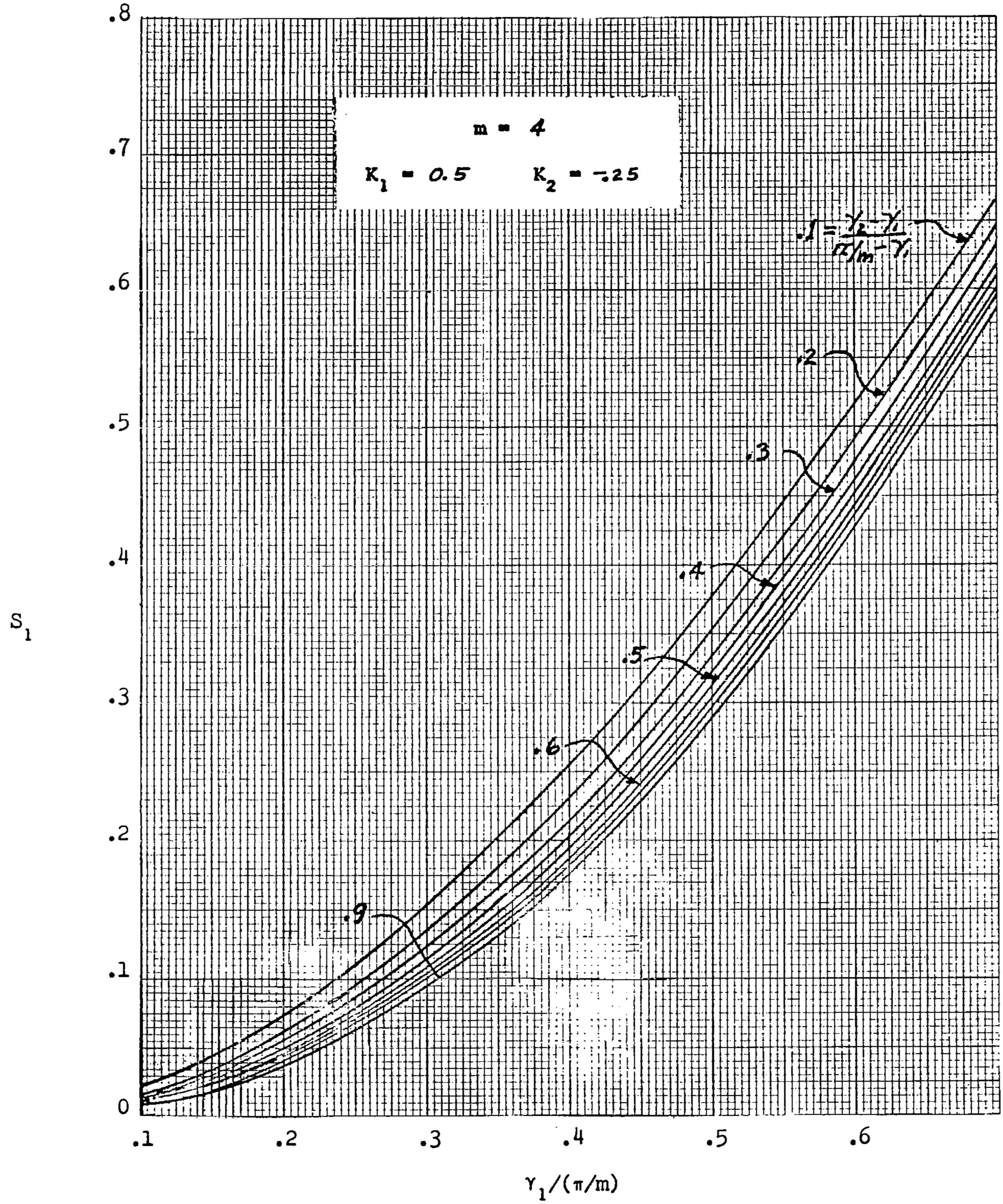


Figure 8.- S_1 versus $\gamma_1/(\pi/m)$ for $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$ = constant.

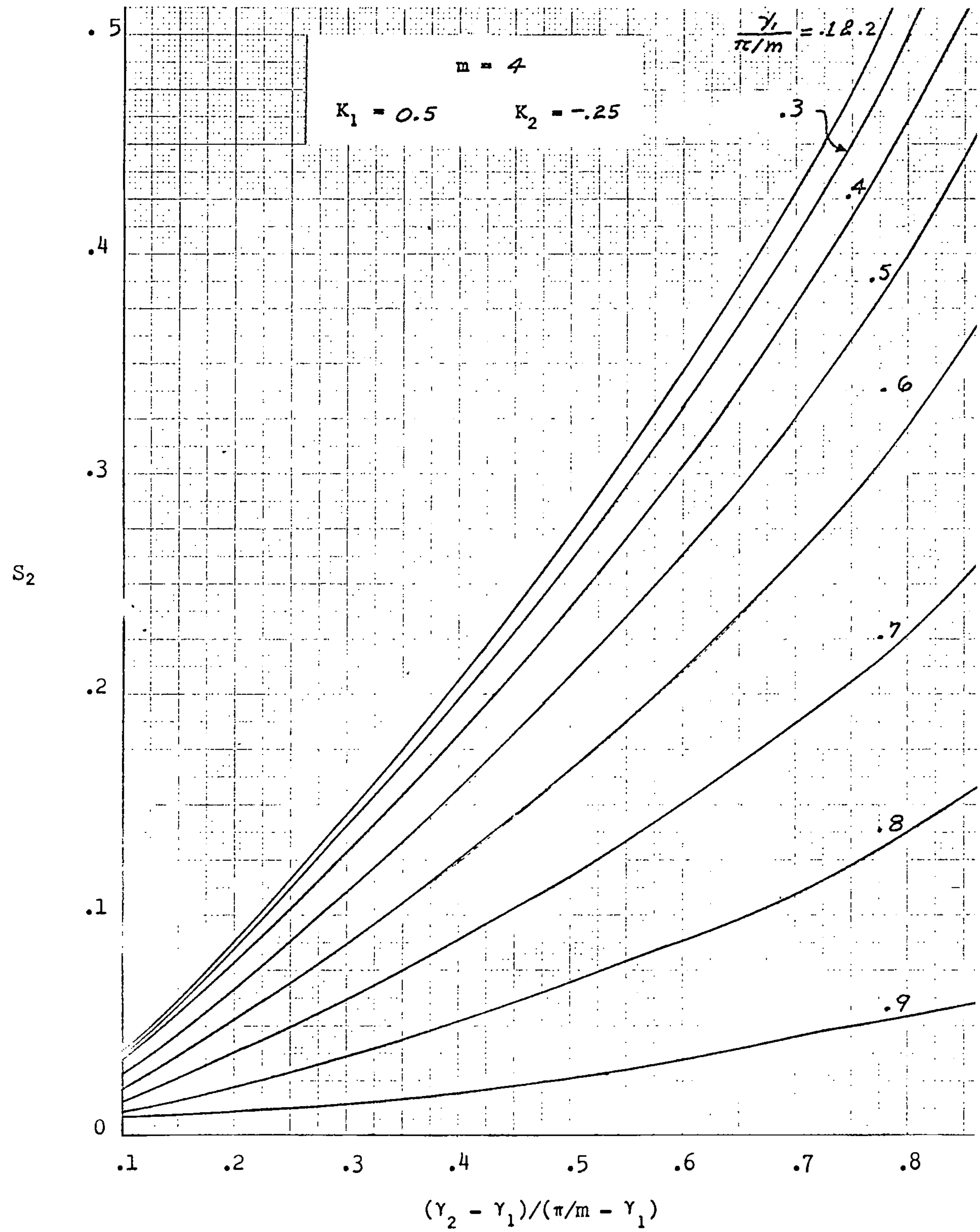


Figure 9.- S_2 versus $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$ for $\gamma_1/(\pi/m)$ = constant.

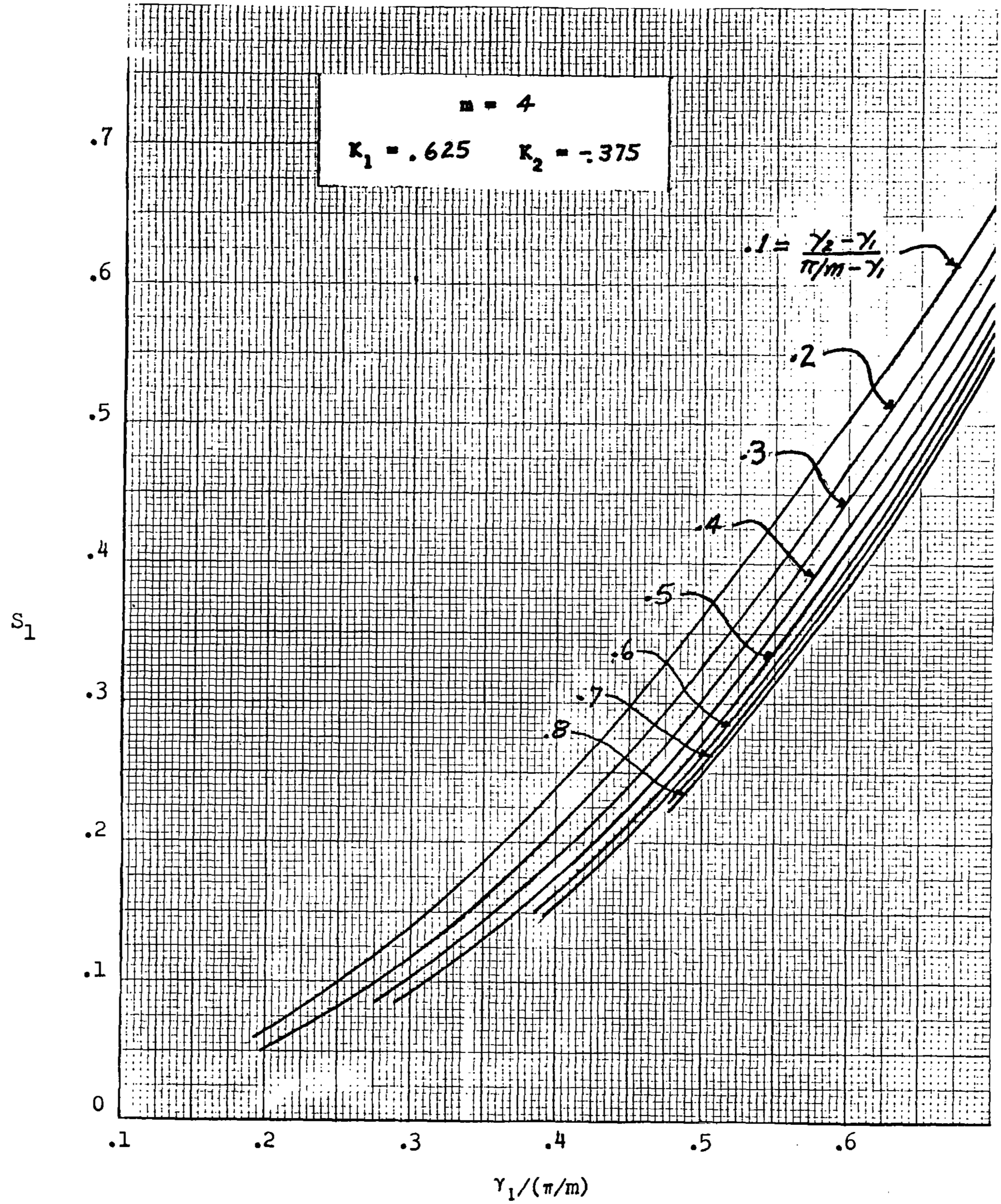


Figure 10.- S_1 versus $\gamma_1/(\pi/m)$ for $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$ = constant.

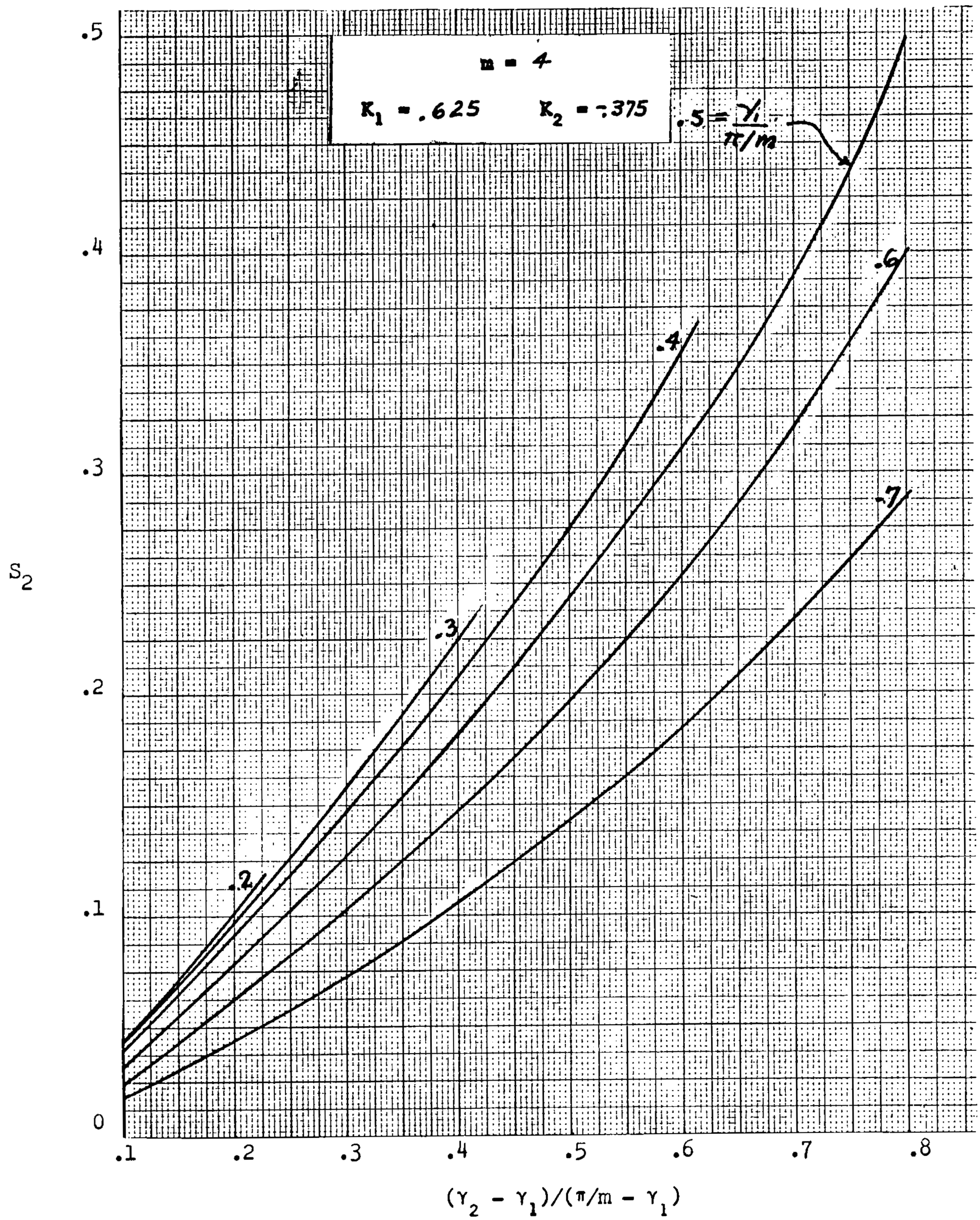


Figure 11.- S_2 versus $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$ for $\gamma_1/(\pi/m)$ = constant.

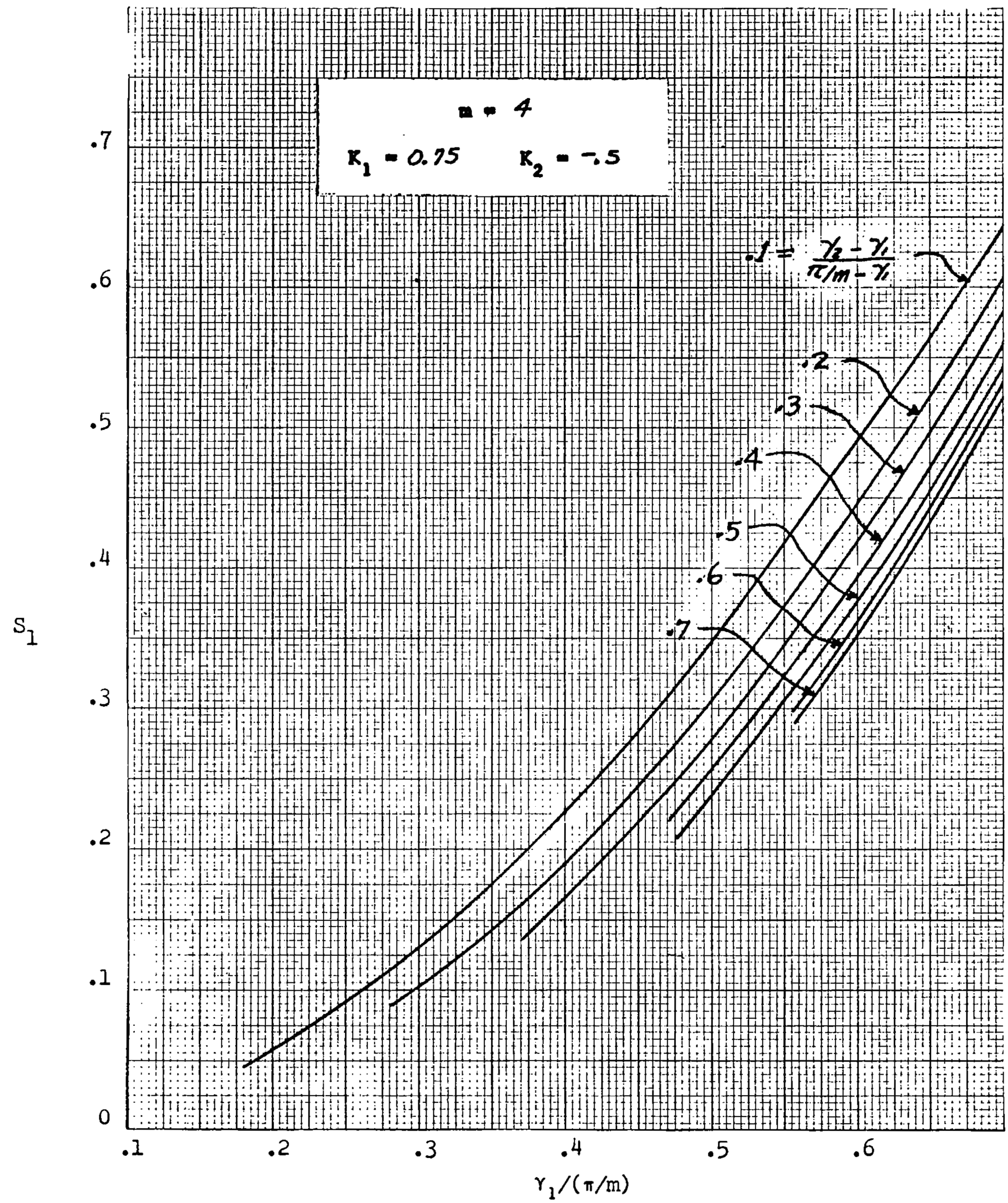


Figure 12.- S_1 versus $\gamma_1/(\pi/m)$ for $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$ = constant.

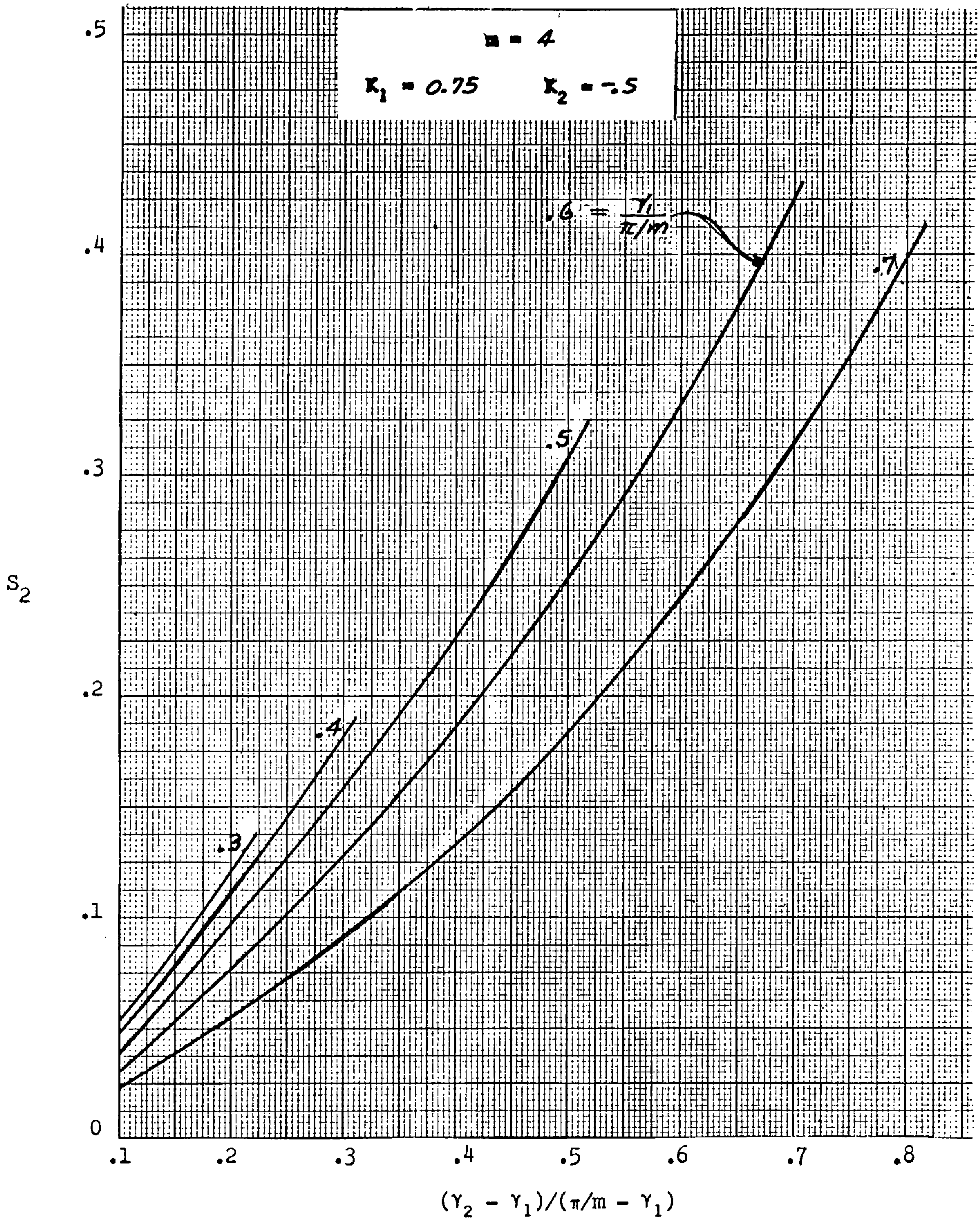


Figure 13.- S_2 versus $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$ for $\gamma_1/(\pi/m)$ = constant.

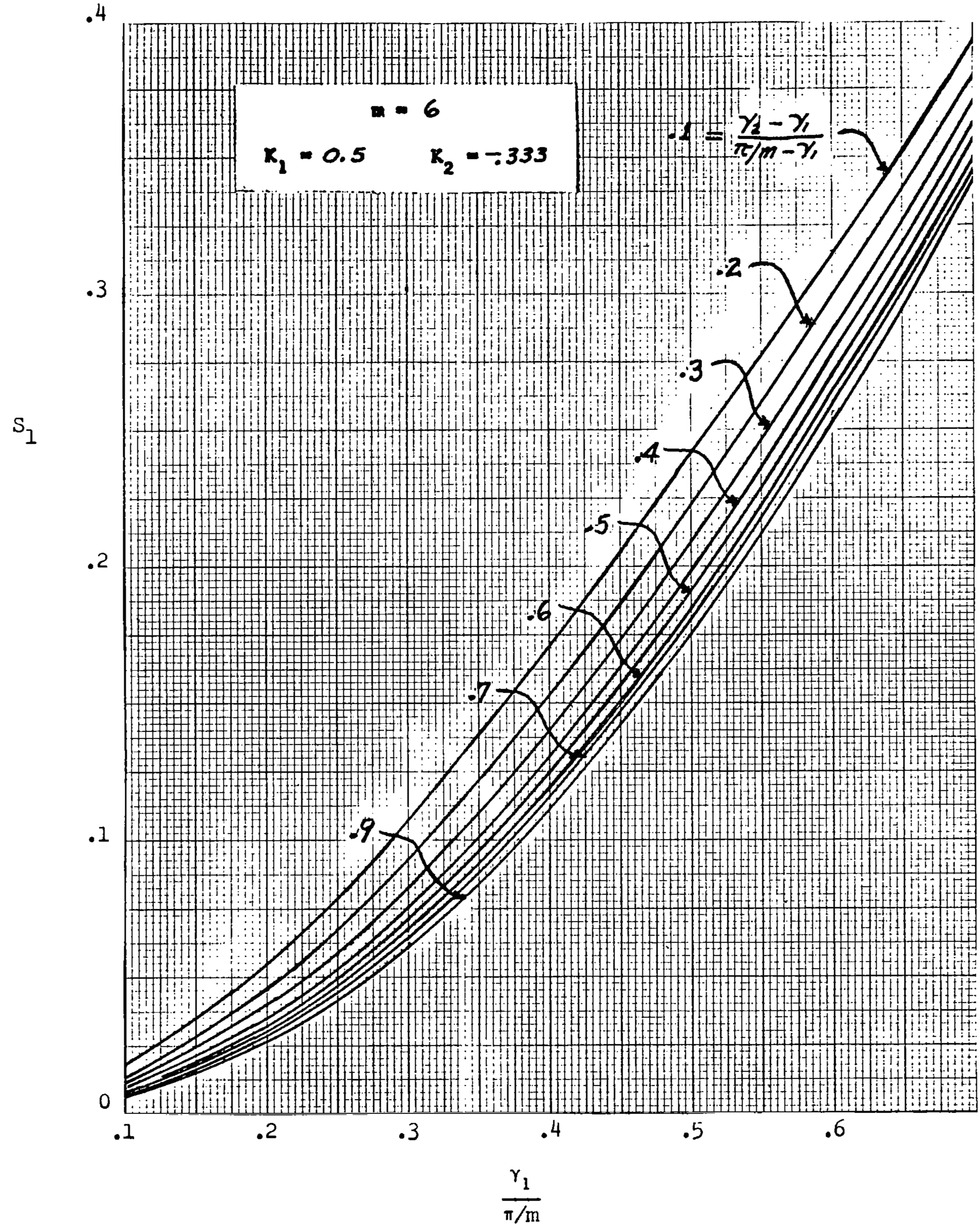


Figure 14.- S_1 versus $\gamma_1/(\pi/m)$ for $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$ = constant.

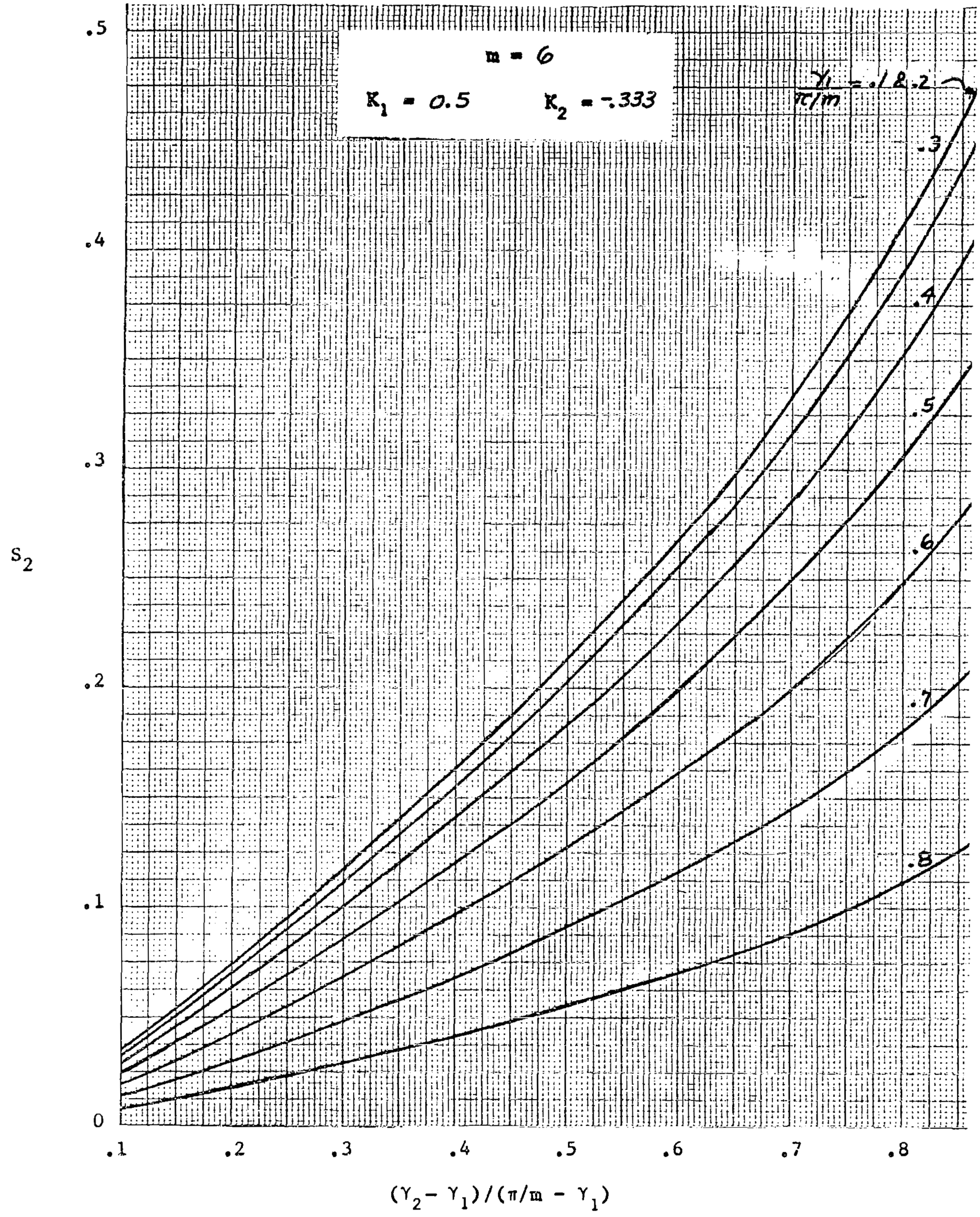


Figure 15.- S_2 Versus $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$ for $\gamma_1/(\pi/m)$ = constant.

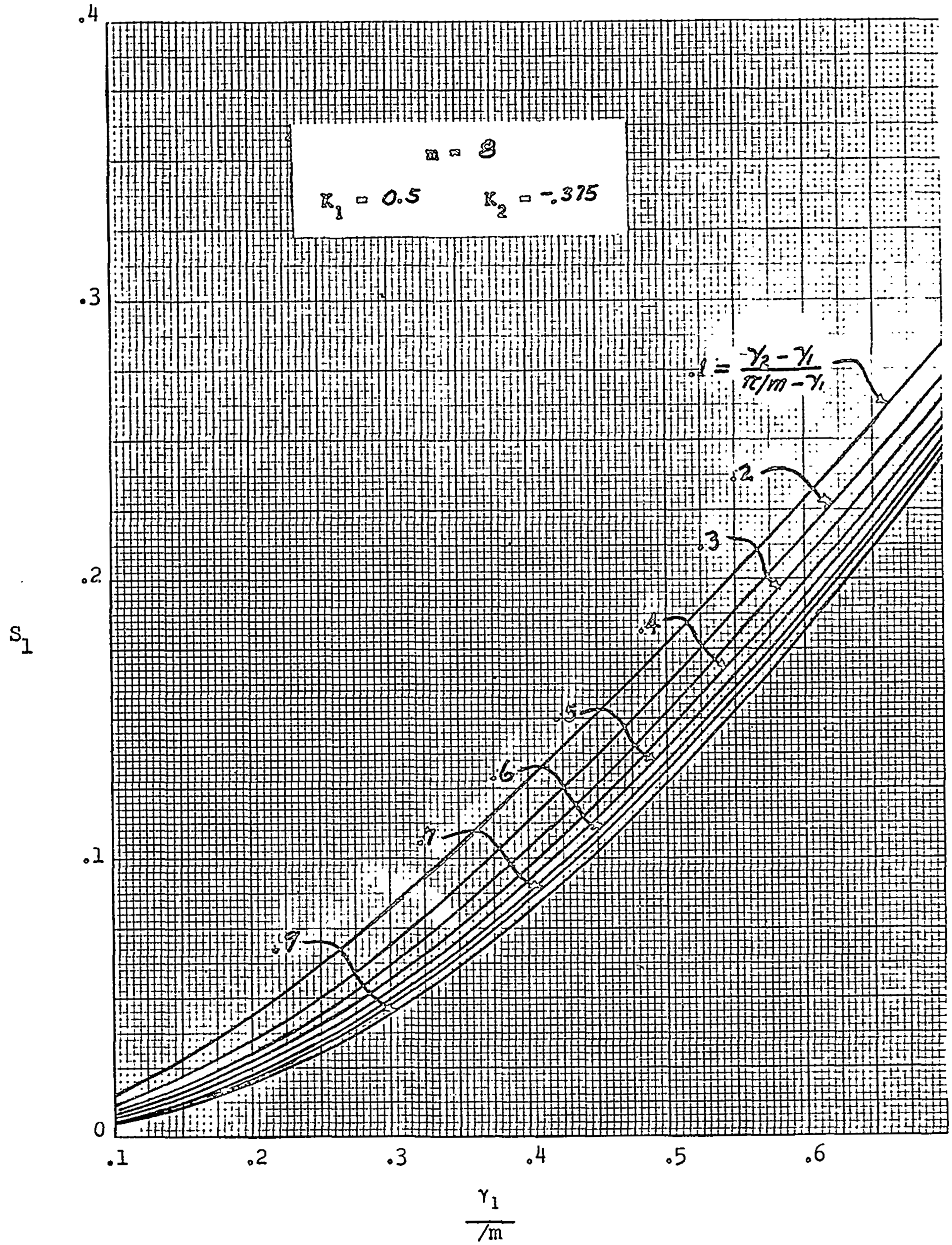


Figure 16.- S_1 Versus $\gamma_1/(\pi/m)$ for $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$ = constant.

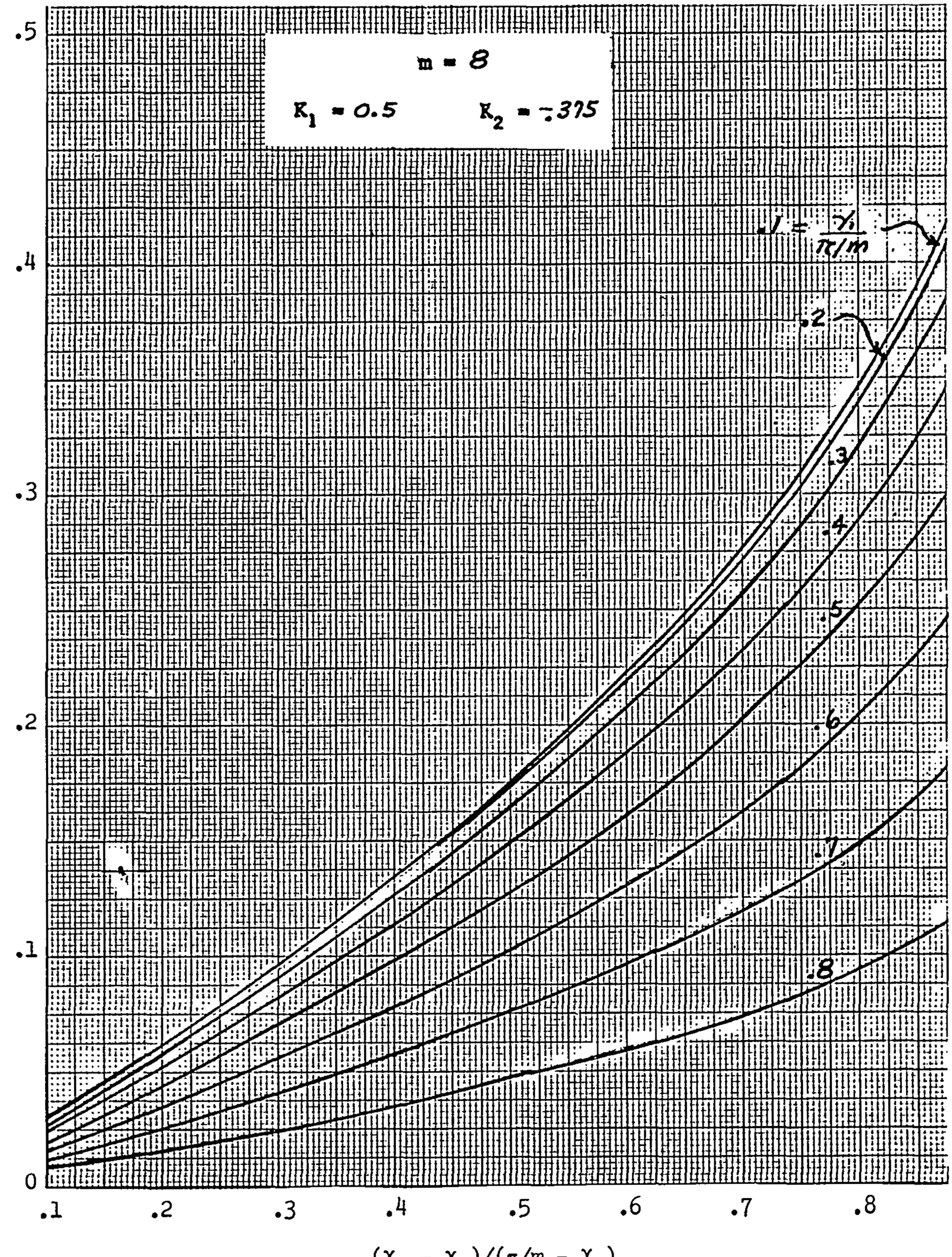
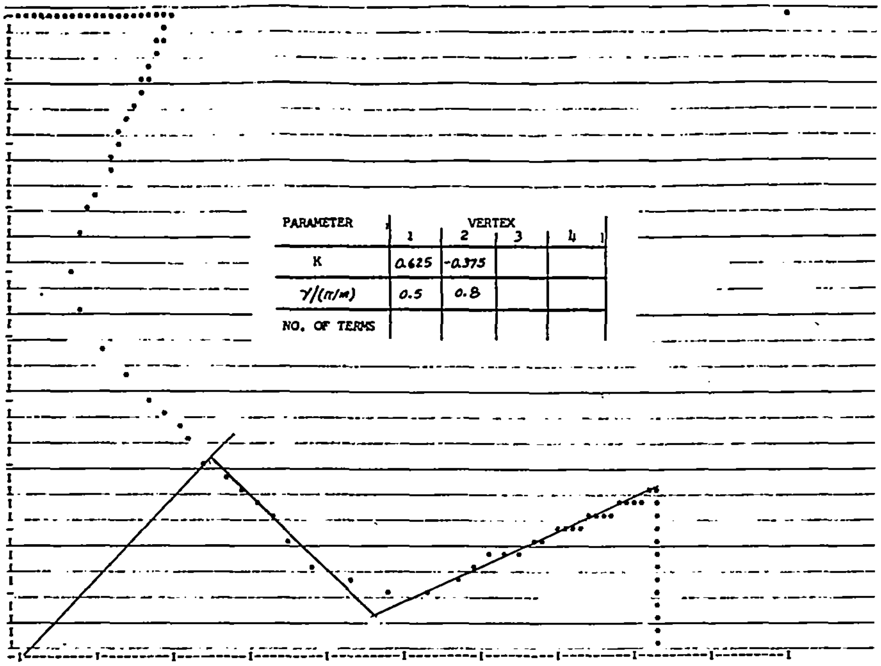


Figure 17.- S_2 Versus $(\gamma_2 - \gamma_1)/(\pi/m - \gamma_1)$ for $\gamma_1/(\pi/m)$ = constant.

APPENDIX B. EXAMPLES OF VARIOUS STARS MAPPED BY EQUATION (5)

The following plots are included to illustrate the type and variety of maps which can be simply obtained from equation (5). Each example was constructed by following the algorithm presented in the section of "CONSTRUCTION OF MAPPING FUNCTIONS."

These plots were made on an IBM paper printer with a capacity of six characters per inch vertically and ten characters per inch horizontally. Thus each point (asterisk) may be in error by $\pm$ 1/12" vertically and $\pm$ 1/20" horizontally, making these plots somewhat crude in places. Hence these plots should not be used to examine the mapping near vertices, but they will give qualitative information on the degree of congruency of the map.



PARAMETER	VERTEX 1	2	3	4
K	0.625	-0.375		
$\gamma/(\pi/m)$	0.5	0.8		
NO. OF TERMS				

Figure 18.- A 4-point second degree star.

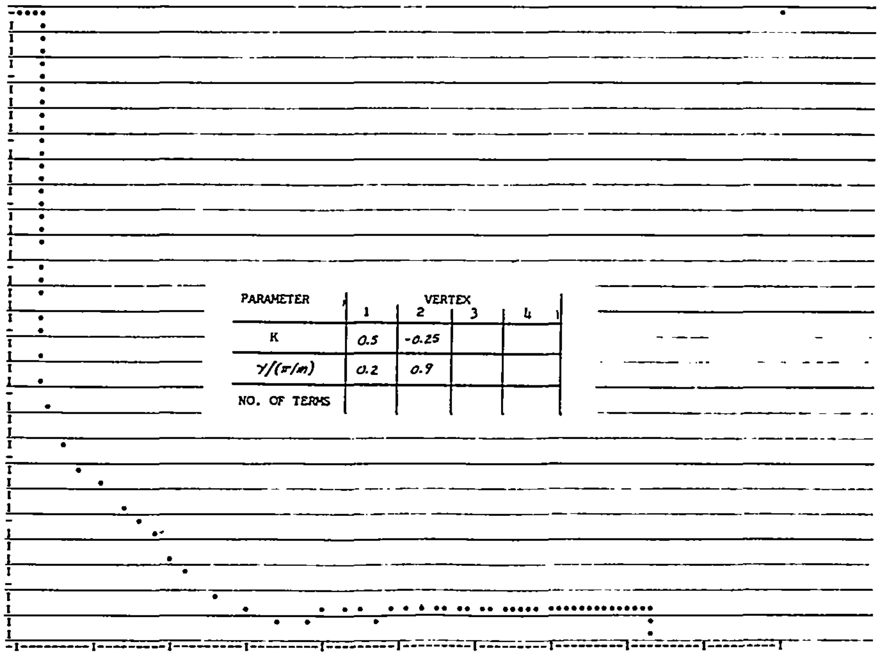


Figure 19.- A 4-point second degree star.

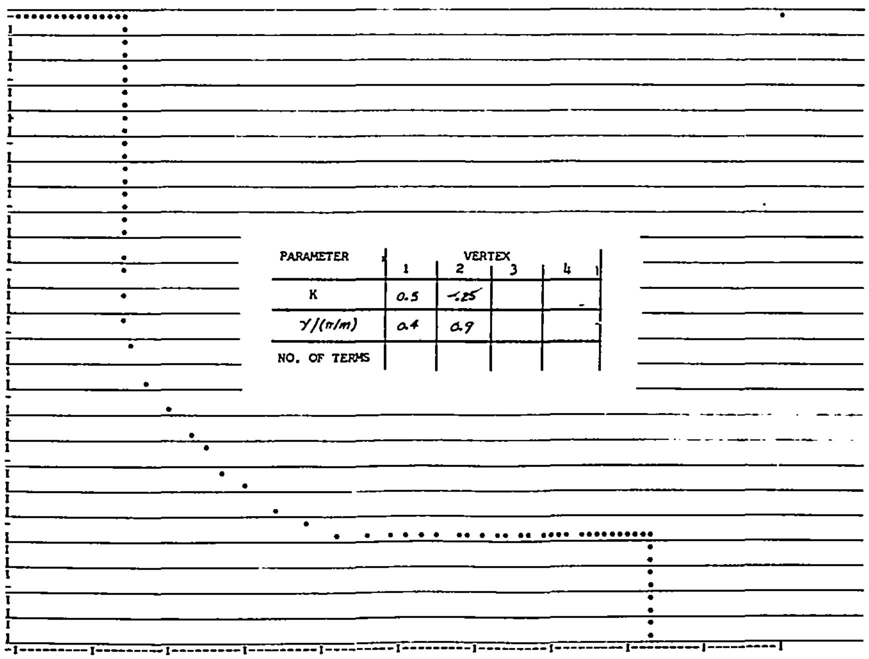


Figure 20.- A 4-point second degree star.

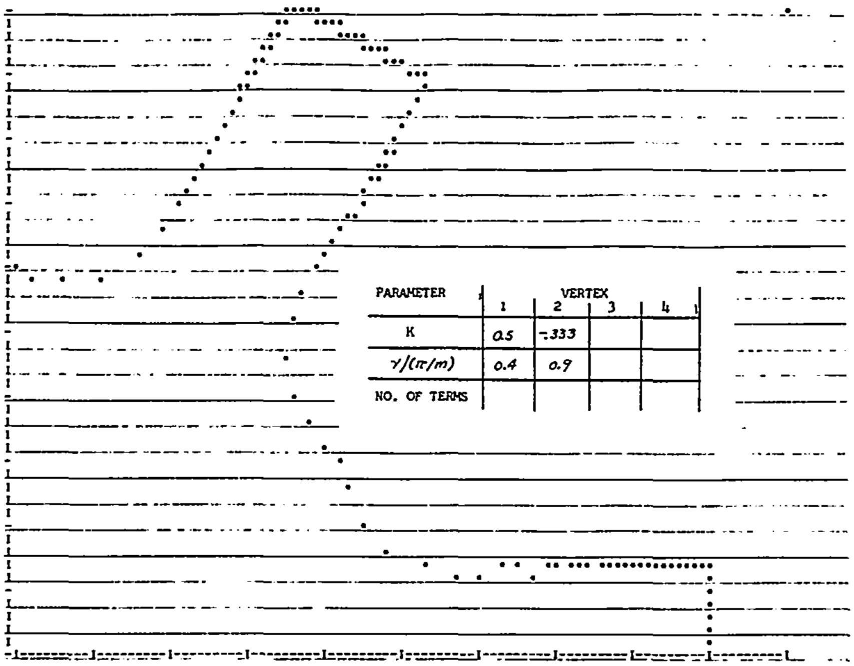


Figure 21.- A 6-point second degree star.

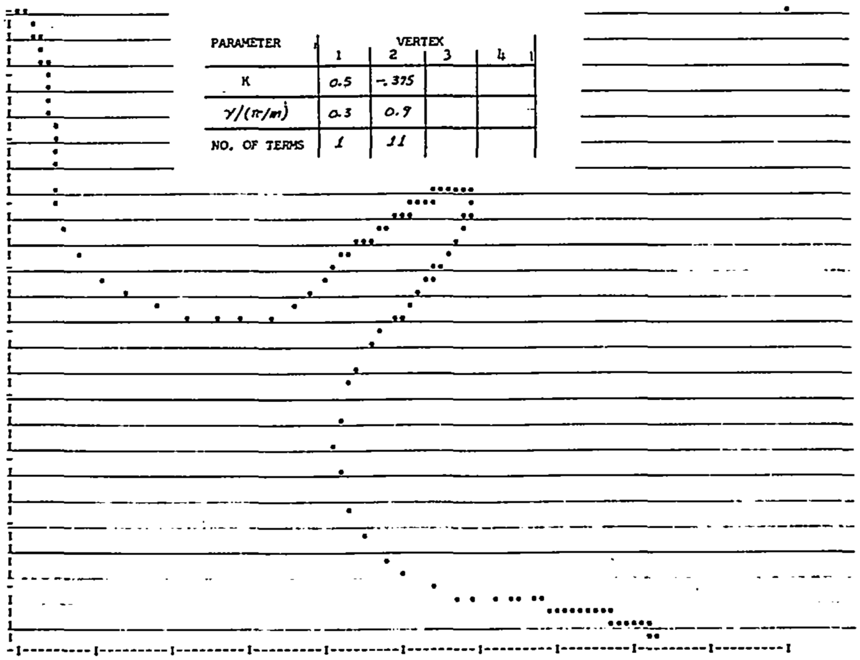


Figure 22.- An 8-point second degree star.

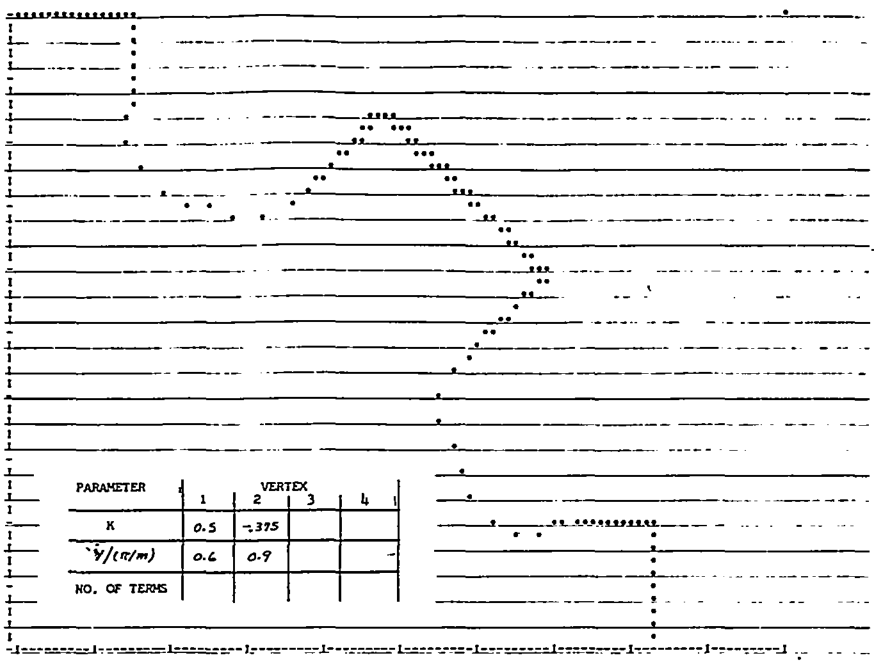


Figure 23.- An 8-point second degree star.

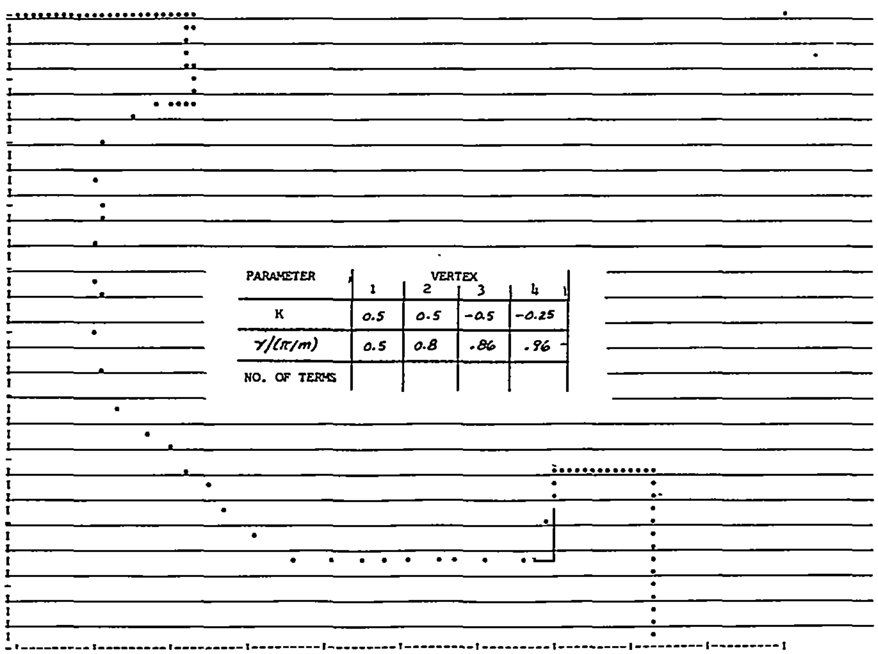
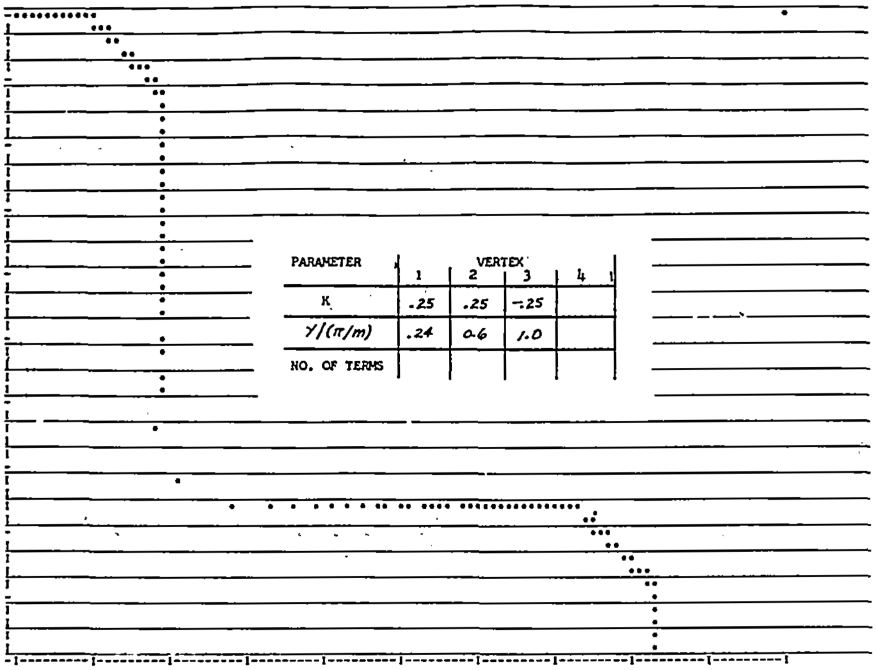
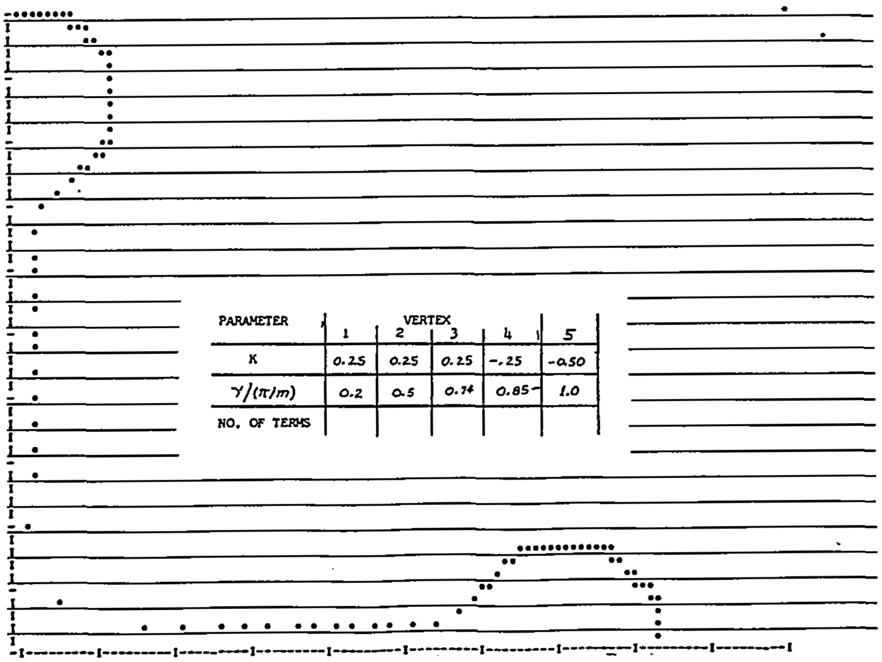


Figure 24.- A 4-point fourth degree star.



PARAMETER	VERTEX 1	2	3	4
K	.25	.25	-.25	
$\gamma/(\pi/m)$	.24	0.6	1.0	
NO. OF TERMS				

Figure 25.- A 4-point second degree star.



PARAMETER	VERTEX 1	2	3	4	5
K	0.25	0.25	0.25	-.25	-0.50
$\gamma/(\pi/m)$	0.2	0.5	0.74	0.85	1.0
NO. OF TERMS					

Figure 26.- A 4-point fourth degree star.

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